MAXIMAL SETS IN α-RECURSION THEORY

BY

M. LERMAN† AND S. G. SIMPSON†, ††

ABSTRACT

Let α be an admissible ordinal, and let α^* be the Σ_1-projectum of α. Call an α-r.e. set M *maximal* if $\alpha - M$ is unbounded and for every α-r.e. set A, either $A \cap (\alpha\text{-}M)$ or $(\alpha\text{-}A) \cap (\alpha\text{-}M)$ is bounded. Call an α-r.e. set M a *maximal subset* of α^* if $\alpha^* - M$ is undounded and for any α-r.e. set A, either $A \cap (\alpha^* - M)$ or $(\alpha^* - A) \cap (\alpha^* - M)$ is unbounded in α^*. Sufficient conditions are given both for the existence of maximal sets, and for the existence of maximal subset *of* α^*. Necessary conditions for the existence of maximal sets are also given. In particular, if $\alpha \geqq \aleph_1^L$ then it is shown that maximal sets do not exist.

0. Introduction

The study of recursive functions on the ordinal numbers was initiated by Takeuti in the late 1950's. Takeuti's concept was generalized by several authors to that of recursive functions on admissible initial segments α of the ordinals. An intensive study of the generalized concept was begun by Sacks in 1964 [6].

The present paper is concerned with generalizations of Friedberg's maximal set theorem to recursion theory on various admissible initial segments of the ordinals. Friedberg's original theorem states that there is a recursively enumerable set (of natural numbers) whose complement is infinite but which cannot be split by a recursively enumerable set into two infinite parts. Such a set is called a *maximal* recursively enumerable set. Kreisel and Sacks [3] proved that there is a metarecursively enumerable set of recursive ordinals whose complement is unbounded but which cannot be split by a metarecursively enumerable set into two unbounded parts. (Actually they proved a somewhat stronger result; see Theorem 2.1

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below.) Sacks [6] made the following observation: let α be the first uncountable cardinal of the constructible universe. Then every unbounded, constructible subset of α can be split by an α-recursive set into two unbounded parts. In particular, for this α, maximal α-recursively enumerable sets do not exist, for any reasonable notion of maximality. In what follows, we prove various existence and nonexistence theorems for maximal α-recursively enumerable sets. Our methods in the proofs of nonexistence significantly extend those of Sacks. Our most quotable result is: if α is an uncountable admissible ordinal, then maximal α-recursively enumerable sets do not exist. (See Theorems 3.5 and 4.4 below.)

The reader will observe that we never settle on a definition of a maximal α-recursively enumerable set. Instead, we make explicit in the statement of each theorem the precise notion of maximality being considered. The main existence and nonexistence results are in Sections 2 and 3. Various subsidiary questions are treated in Sections 4 and 5. The paper ends with a list of open problems in Section 6.

1. Preliminaries

The reader should not read this section through but rather refer to it as needed.

We use von Neumann's definition of ordinal. Thus an ordinal is identified with the set of all smaller ordinals. Our set-theoretical notation is standard. In particular $\cup$ (union), $\cap$ (intersection), $\times$ (Cartesian product), $''$(range), $-$ (set-theoretic difference), $\subseteq$ (subset of), $\in$ (element of), and $\emptyset$ (empty set) have their usual meanings.

As in Gödel [1], we define the *constructible hierarchy*: $M_0 = \{\emptyset\}$; $M_{\xi+1} = \{X \subseteq M_\xi \mid X$ is first order definable over $\langle M_\xi, \in \rangle$ allowing parameters from $M_\xi\}$; $M_\lambda = \cup \{M_\xi \mid \xi < \lambda\}$ for limit ordinals λ. The constructible universe is defined by: $L = \cup \{M_\xi \mid \xi$ is an ordinal$\}$.

Throughout this paper, α is a fixed but arbitrary admissible ordinal. Lower case Greek letters denote ordinals less than α except for β which denotes a limit ordinal less than or equal to α. A set $X \subseteq \beta$ is said to be *unbounded in* β if $\cup X = \beta$; otherwise, it is said to be *bounded below* β. We sometimes write *unbounded* for unbounded in α and *bounded* for bounded below α.

A partial function from α into α is said to be *α-recursive* if its values can be calculated via an equation calculus resembling Kleene's, but allowing infinitary bounded quantifications such as $(\exists x < \delta) \cdots$ where $\delta < \alpha$. (For further details, see

[6].) A subset of α is said to be *α-recursive* if its characteristic function is α-recursive. A subset of α is said to be *α-recursively enumerable* (abbreviated α-r.e.) if it is the domain or range of an α-recursive partial function. Every α-r.e. set can be written as the range of a one-one α-recursive function whose domain is an ordinal less than or equal to α.

A subset of α is said to be *α-finite* if it is α-recursive and bounded. It can be shown that a subset of α is α-finite if and only if it is a member of M_α. A basic principle of α-recursion theory is: if f is an α-recursive partial function and $K \subseteq \mathrm{dom}(f)$ is α-finite, then $f''K$ is α-finite.

Following Rogers [5, pp. 301–307], we define the *α-arithmetical hierarchy*. Thus, a relation on α is Σ_0 if it is α-recursive, Π_n if its complement is Σ_n, and Σ_{n+1} if it is the projection of a Π_n relation. In particular, a relation is Σ_1 if and only if it is α-r.e. For $n \geqq 1$, it can be shown that a relation is Σ_n if and only if it is Σ_n definable over $\langle M_\alpha, \in \rangle$ allowing parameters from M_α, in the sense of Lévy [4]. A partial function on α is said to be Σ_n if its graph is Σ_n. In particular, a partial function is Σ_1 if and only if it is α-recursive.

Warning: the bounded quantifier manipulations of Rogers [5, p. 311] do not generalize to α-recursion theory except in very special circumstances. However Jensen [2] has proved the following remarkable theorem.

THEOREM 1.1 (Jensen). *For $n \geqq 1$, every Σ_n relation on α can be uniformized, by a Σ_n partial function.*

Following Jensen, we define for $n \geqq 1$ the *Σ_n projectum* of α to be the least β such that there is a Σ_n partial function with domain a subset of β and range α.

THEOREM 1.2 (Jensen). *For $n \geqq 1$, the Σ_n projectum of α is equal to the least β such that there is a Σ_n subset of β which is not α-finite.*

The Σ_1 projectum of α is sometimes denoted α^*. If K is an α-finite set, the *α-cardinality* of K is the least γ such that there is an α-finite one-one correspondence between K and γ. An ordinal less than α is an *α-cardinal* if it is equal to its own α-cardinality. An α-cardinal γ is *regular* if no α-finite subset of γ is unbounded in γ but of order type less than γ. The following facts are easily verified.

1. If α^* is less than α, then α^* is the largest α-cardinal.
2. Every α-cardinal is either regular or a limit of regular α-cardinals.

If $n \geqq 1$ and β is a limit ordinal less than or equal to α, we define the *Σ_n cofinality* of β to be the least ordinal λ such that there is a Σ_n function with domain λ and range unbounded in β. Thus, for example, the Σ_1 cofinality of α is just α. For $n \geqq 2$, the Σ_n cofinality of α^* is equal to the Σ_n cofinality of α.

The end of a proof will be indicated by $\square$.

2. An existence result

Our first theorem says that maximal α-r.e. sets exist for a wide class of admissible ordinals α.

THEOREM 2.1. *Suppose there is a Σ_2 function f with domain ω and range α. Then there is an* α-r.e. *set M such that $\alpha - M$*

i) *is unbounded in α;*

ii) *has order type ω;*

iii) *cannot be split by an α-r.e. set into two infinite parts.*

Our proof follows Kreisel and Sacks' [3] construction of a maximal meta-r.e. set.

Proof. We adopt an α-recursive simultaneous enumeration of the α-r.e. sets. Thus $\langle R^\sigma_\rho \mid \sigma < \alpha \;\&\; \rho < \alpha \rangle$ is an α-recursive double sequence of α-finite sets, R^σ_ρ is nondecreasing as a function of σ, and $R_\rho = \cup \{R^\sigma_\rho \mid \sigma < \alpha\}$ ranges over the α-r.e. sets as ρ ranges over α.

Let $f(\sigma, n)$ be an α-recursive function such that $f(n) = \lim_\sigma f(\sigma, n)$ for each finite n .Such an α-recursive approximation to f exists because f is Σ_2.

We shall define functions $v(\sigma, e)$ $(e < \omega)$ and M^σ in an α-recursive manner by induction on σ. The sequence $\langle M^\sigma \mid \sigma < \alpha \rangle$ will be nondecreasing. At the end of the construction, we shall put $M = \cup \{M^\sigma \mid \sigma < \alpha\}$ and prove that $\alpha - M$ has the desired properties (i)–(iii).

As a preliminary to stage σ of the construction, we put

$$M^{<\sigma} = \cup \{M^\tau \mid \tau < \sigma\}.$$

If $\alpha > \omega$, then $M^{<\sigma}$ will be α-finite. In any case, $\alpha - M^{<\sigma}$ will be unbounded.

Stage σ: For each $\eta < \alpha$, we say η is in the jth *e-state* at stage σ if $\eta \notin M^{<\sigma}$ and

$$j = \Sigma \{2^{e-i} \mid \eta \in R^\sigma_{f(\sigma,i)} \;\&\; i \leqq e\}.$$

Note that the number of e-states is finite and that each e-state except the 0th is α-finite at stage σ. Define $v(\sigma, e)$ $(e < \omega)$ by induction on e as follows. Since $\alpha - M^{<\sigma}$ is unbounded, there is an e-state which contains an η exceeding every member of

(*) $$\{v(\sigma, i) \mid i < e\} \cup \{f(\tau, i) \mid \tau \leqq \sigma \,\&\, i \leqq e\}.$$

Let $j(\sigma, e)$ be the highest such e-state. Let $v(\sigma, e)$ be the least η which exceeds every member of (*) and is in the $j(\sigma, e)$th e-state. Define M^σ by

$$M^\sigma = \{\mu \mid \mu < v(\sigma, 0) \vee \exists n(v(\sigma, n) < \mu < v(\sigma, n+1))\}.$$

This completes stage σ of the construction.

Note that $M^{<\sigma} \subseteq M^\sigma$ and that $\langle v(\sigma, n) \mid n < \omega\rangle$ are the first ω members of $\alpha - M^\sigma$ in increasing order.

LEMMA. 2.2. *For each* e, $v(e) = \lim_\sigma v(\sigma, e)$ *exists, i.e.* $\exists\sigma\forall\tau \geqq \sigma(v(\tau, e) = v(\sigma, e))$

PROOF. We argue by induction on e. Suppose that $v(i) = \lim_\sigma v(\sigma, i)$ exists for each $i < e$. Let γ be the least ordinal exceeding every member of

$$\{v(i) \mid i < e\} \cup \{f(\sigma, i) \mid \sigma < \alpha \,\&\, i \leqq e\}.$$

Let σ_0 be such that $\forall \sigma \geqq \sigma_0$ $(\forall i < e\ (v(\sigma, i) = v(i)) \,\&\, \forall i \leqq e\ (f(\sigma, i) = f(i)))$. Then at any stage $\sigma \geqq \sigma_0$, $j(\sigma, e)$ is the highest e-state containing an $\eta \geqq \gamma$. Let j be the largest member of $\{j(\sigma, e) \mid \sigma \geqq \sigma_0\}$. Let δ be the least $\eta \geqq \gamma$ such that η is in the jth e-state at some stage $\sigma \geqq \sigma_0$. Let $\sigma_1 \geqq \sigma_0$ be such that δ is in the jth e-state at stage σ_1. Then $j(\sigma_1, e) = j$ and $v(\sigma_1, e) = \delta$. Hence by induction, $j(\sigma, e) = j$ and $v(\sigma, e) = \delta$ for all $\sigma \geqq \sigma_1$. □

Recall that $f(\sigma, e) < v(\sigma, e) < v(\sigma, e+1)$ for all σ and e, and that the range of f is unbounded. It follows that $\alpha - M = \{v(e) \mid e < \omega\}$ is unbounded and has order type ω.

LEMMA 2.3. $\alpha - M$ *cannot be split by an* α*-r.e. set into two unbounded parts.*

PROOF. Suppose not. Let e be the least n such that $R_{f(n)}$ splits $\alpha - M$ into two unbounded parts. Then there are c, d, i, j such that $e \leqq c < d$, $i < j$, and $v(c)$ (resp. $v(d)$) is in the ith (resp. jth) e-state at all sufficiently large stages σ. Let σ_0 be such that $\forall\sigma \geqq \sigma_0\ \forall k \leqq d(v(\sigma, k) = v(k) \,\&\, f(\sigma, k) = f(k))$. Let $\sigma \geqq \sigma_0$ be such that $v(c)$ (resp. $v(d)$) is in the ith (resp. jth) e-state at stage σ. Let i^* (resp. j^*) be the c-state of $v(c)$ (resp. $v(d)$) at stage σ. Then $v(d)$ exceeds every member of

$$\{v(\sigma, k) \mid k < c\} \cup \{f(\tau, k) \mid \tau \leqq \sigma \,\&\, k \leqq c\}$$

so $j(\sigma, c) \geqq j^* > i^*$. On the other hand, $v(\sigma, c) = v(c)$ so $j(\sigma, c) = i^*$, a contradiction. □

The proof of Theorem 2.1 is complete.

There are many interesting examples of admissible ordinals α satisfying the hypothesis of Theorem 2.1. In the first place, the hypothesis is satisfied if $\alpha^* = \omega$. In this case, Theorem 2.1 specializes to the earlier result of Kreisel and Sacks. Another class of examples is provided by the following theorem whose proof appears in the second author's Ph.D. thesis [7].

THEOREM 2.4. *Let F be a Σ_4 sentence of the ZF language. Suppose α is the least admissible ordinal such that $\langle M_\alpha, \in \rangle$ satisfies F. Then there is a Σ_2 function with domain ω and range α.*

To be specific, consider the following examples.

1. Let α be the least admissible ordinal greater than ω such that $\alpha^* = \alpha$.

2. Let α be the least admissible ordinal such that $\omega < \alpha^* < \alpha$.

3. Let α be the least admissible ordinal greater than ω such that $\langle M_\alpha, \in \rangle$ satisfies the power set axiom.

It is easy to construct Σ_4 sentences showing that each of these α's falls under the purview of Theorem 2.4. Hence, for each of these α's, maximal α-r.e. sets exist by Theorem 2.1.

3. Some nonexistence results

In this section, we present some theorems to the effect that for certain admissible ordinals α, maximal α-r.e. sets do not exist.

LEMMA 3.1. *Let λ be the Σ_2 cofinality of α. There is a sequence of sets $\langle H_\xi \mid \xi < \lambda \rangle$ such that*

i) $\forall \xi < \eta < \lambda \, (H_\xi \cap H_\eta = \varnothing)$;

ii) $\alpha = \cup \{H_\xi \mid \xi < \lambda\}$;

iii) *the sets H_ξ, $\xi < \lambda$, are simultaneously α-r.e.*;

iv) $\forall \eta < \lambda \, (\cup \{H_\xi \mid \xi < \eta\}$ *is α-finite*).

PROOF. If $\lambda = \alpha$, the lemma is trivial so assume $\lambda < \alpha$. Let f be a Σ_2 function with domain λ and range an unbounded subset of α. Let $f(\sigma, \xi)$ be an α-recursive function such that for all $\xi < \lambda$, $f(\xi) = \lim_\sigma f(\sigma, \xi)$, i.e. $\forall \xi < \lambda \; \exists \sigma \forall \tau \geqq \sigma (f(\tau, \xi) = f(\xi))$. Let us say that $f(\xi)$ *changes value at stage* σ if $\forall \sigma' < \sigma \exists \tau \; (\sigma' \leqq \tau < \sigma \, \& \, f(\tau, \xi) \neq f(\sigma, \xi))$. For each τ, let $n(\tau)$ be the least $\sigma \geqq \tau$ such that some $f(\xi)$ changes value at stage σ. This $n(\tau)$ exists since otherwise $f(\xi) = f(\tau, \xi)$ for all $\xi < \lambda$ which would imply that f has bounded range. Put τ into H_η if η is the least ξ such that $f(\xi)$ changes value at stage $n(\tau)$. Properties (i)-(iii) are obvious. Property (iv) holds because otherwise there would be a Σ_2 function with domain η and range unbounded in α. □

THEOREM 3.2. *Assume that the Σ_2 cofinality of α is less than the Σ_2 projectum of α. Then every unbounded Π_1 set can be split by an α-recursive set into two unbounded parts.*

PROOF. Let λ and $\langle H_\xi \mid \xi < \lambda \rangle$ be as in Lemma 3.1. Let S be an unbounded Π_1 set. Define a set $X \subseteq \lambda$ by putting $\eta \in X$ if and only if

$$(\exists \gamma \in H_\eta \cap S)(\cup \{H_\xi \cap S \mid \xi < \eta\} \subseteq \gamma).$$

Clearly X is Σ_2. Since λ is less than the Σ_2 projectum, it follows by Theorem 1.2 that X is α-finite. Also, X is unbounded in λ, since S is unbounded in α. (Here we are using property (iv) in the statement of Lemma 3.1.) Hence, X can be split into two α-finite sets, X_0 and X_1, each of which is unbounded in λ. Put $B_0 = \cup \{H_\xi \mid \xi \in X_0\}$ and $B_1 = \cup \{H_\xi \mid \xi \in X_1\}$. Then B_0 and B_1 are disjoint and α-recursive. Furthermore, $B_0 \cap S$ and $B_1 \cap S$ are unbounded. Thus B_0 splits S into two unbounded parts. □

As an example of an interesting admissible ordinal to which Theorem 3.2. applies, we may take $\alpha = \omega_\omega^L$, the ωth infinite cardinal of the constructible universe. The function $\langle \omega_n^L \mid n < \omega \rangle$ is then Σ_2 so α is Σ_2 cofinal with ω. On the other hand, as a cardinal of L, α is clearly equal to its own Σ_2 projectum. Thus, for this α, there are no maximal α-r.e. sets.

Let B be an α-r.e. set. Write $B = \cup \{B^\sigma \mid \sigma < \alpha\}$ where $\langle B^\sigma \mid \sigma < \alpha \rangle$ is an α-recursive nondecreasing sequence of α-finite sets. Let $\gamma < \alpha$ be fixed. Clearly the order type of $\gamma - B^\sigma$ is nonincreasing, hence eventually constant, as a function of σ. The following simple observation requires proof.

LEMMA 3.3. *For each $\gamma < \alpha$ there is σ such that $\gamma - B^\sigma$ has the same order type as $\gamma - B$.*

PROOF. Consider the least γ for which the lemma fails. It is easy to see that γ is a limit ordinal and that $\gamma - B$ is unbounded in γ. Let θ be the order type of $\gamma - B$. For each σ, let $f(\sigma)$ be the supremum of the first θ elements of $\gamma - B^\sigma$. Clearly, $f: \alpha \to \gamma$ is α-recursive and nondecreasing. Furthermore, $\gamma = \lim f$ in the weak sense that

$$\forall \gamma' < \gamma \exists \sigma' \forall \sigma \geqq \sigma' \ (f(\sigma) \geqq \gamma').$$

Now for each $\nu < \gamma$, let $g(\nu)$ be the least σ such that $f(\sigma) \geqq \nu$ Then $g: \gamma \to \alpha$ is α-recursive and unbounded. This contradicts the admissibility of α. □

THEOREM 3.4. *Let S be an unbounded Π_1 set which cannot be split by a Π_1 set into two unbounded parts. Let μ be a limit ordinal such that every final segment of S has order type greater than μ. Then there is a Σ_3 function $f: \mu \to \alpha$ such that $\{\xi < \mu \mid f(\xi) < \gamma\}$ is finite for all $\gamma < \alpha$.*

PROOF. Write $\alpha - S = M = \cup \{M^\sigma \mid \sigma < \alpha\}$ where $\langle M^\sigma \mid \sigma < \alpha \rangle$ is an α-recursive nondecreasing sequence of α-finite sets. For each $\xi < \mu$, put

$$A_\xi = \{\gamma \mid \exists\sigma\exists\eta\, (\gamma - M^\sigma \text{ has order type } \mu \cdot \eta + \xi)\}.$$

Then A_ξ is α-r.e. and by Lemma 3.3, $S \cap A_\xi$ is unbounded. Hence $S - A_\xi$ is bounded. The relation $S - A_\xi \subseteq \gamma$ is clearly Π_2. Hence by Jensen's Uniformization Theorem 1.1, there is a Σ_3 function $f: \mu \to \alpha$ such that $S - A_\xi \subseteq f(\xi)$ for each $\xi < \mu$. But for each $\gamma < \alpha$, $\{\text{order type of } \gamma - M^\sigma \mid \sigma < \alpha\}$ is finite. Hence $\{\xi < \mu \mid f(\xi) < \gamma\}$ is finite. □

COROLLARY. *Suppose there is an unbounded Π_1 set which cannot be split by a Π_1 set into two unbounded parts. Then α is Σ_3 cofinal with ω.*

PROOF. Let S be such a Π_1 set. If S has a final segment of order type ω, then in fact α is Σ_2 cofinal with ω. If not, apply Theorem 3.4 with $\mu = \omega$ to show that α is Σ_3 cofinal with ω. □

Our next theorem says that maximal α-r.e. sets do not exist for uncountable admissible ordinals α.

THEOREM 3.5. *Assume α is greater than or equal to ω_1^L, the first uncountable cardinal of the constructible universe. Then every unbounded Π_1 set can be split by a Π_1 set into two unbounded parts.*

PROOF. Suppose for contradiction that $\alpha \geqq \omega_1^L$ and S is an unbounded Π_1 set which cannot be split by a Π_1 set into two unbounded parts.

Case I. S has a final segment of order type less than ω_1^L. Then the Σ_2 cofinality of α is less than ω_1^L (in fact it is ω) which in turn is less than or equal to the Σ_2 projectum of α. Hence Theorem 3.2 provides a contradiction.

Case II. S has a final segment of order type ω_1^L. This contradicts the corollary to Theorem 3.4.

Case III. Not Case I or II. By Theorem 3.4 with $\mu = \omega_1^L$, we obtain a Σ_3

function $f\colon \omega_1^L \to \alpha$ such that $\{\xi < \omega_1^L \mid f(\xi) < \gamma\}$ is finite for each $\gamma < \alpha$. Hence ω_1^L is constructibly countable, a contradiction. □

The hypothesis $\alpha \geqq \omega_1^L$ in Theorem 3.5 is much stronger than necessary. It could, for example, be replaced by the weaker hypothesis that α is uncountable in $M_{\alpha+}$ where α^+ is the next admissible ordinal after α. This is because Theorem 3.5, and indeed all the results in this paper, can be proved in Kripke-Platek set theory. The question of how much farther the hypothesis of Theorem 3.5 can be weakened, will be answered completely in a future paper (see footnote in Section 6)'

4. Maximal subsets of α^*

In [3], Kreisel and Sacks considered maximal Π_1^1 subsets of ω. This suggests that we now study maximal α-r.e. subsets of α^* in case $\alpha^* < \alpha$. The only known existence result here is the following, due essentially to Kreisel and Sacks [3]:

THEOREM 4.1. *If $\alpha^* = \omega$, then there is an α-r.e. subset of ω, M, such that $\omega - M$ is infinite but cannot be split by an α-r.e. set into two infinite parts.*

PROOF. Since $\alpha^* = \omega$, there is an α-recursive partial function p with domain a subset of ω and range α. Let $A(\sigma,i)$ be an α-recursive predicate such that $p(i)$ is defined if and only if $(\exists\sigma)A(\sigma, i)$. Define

$$P_i^\sigma = \begin{cases} R_{p(i)}^\sigma \cap \omega & \text{if } (\exists\tau \leqq \sigma)A(\tau, i); \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus $\langle P_i^\sigma \mid \sigma < \alpha \,\&\, i < \omega\rangle$ is an α-recursive double sequence of α-finite subsets of ω; P_i^σ is nondecreasing as a function of σ; and $P_i = \cup\{P_i^\sigma \mid \sigma < \alpha\}$ ranges over the α-r.e. subsets of ω as i ranges over ω.

Modify the proof of Theorem 2.1 as follows. For each $n < \omega$ say n is in the jth e-state if $n \notin M^{<\sigma}$ and $j = \Sigma\{2^{e-i} \mid n \in P^\sigma \,\&\, i \leqq e\}$. Replace (*) by $\{v(\sigma, i) \mid i < e\}$. (Thus the function f plays no role in the modified construction.) Change Lemma 2.2 to read: for each $e < \omega$, $v(e) = \lim_\sigma v(\sigma, e)$ is finite. Change Lemma 2.3 to read: $\omega - M$ cannot be split by an α-r.e. set into two infinite parts. The proofs of the modified lemmas go through virtually unchanged. □

The following general lemma leads to a nonexistence result for maximal α-r.e subsets of α^*.

LEMMA 4.2. *Let S be a bounded Π_1 set of order type less than α^*. Then S is α-finite.*

PROOF. Let $S \subseteq \gamma < \alpha$. Write $\gamma - S = M = \cup\{M^\sigma \mid \sigma < \alpha\}$ where $\langle M^\sigma \mid \sigma < \alpha\rangle$

is an α-recursive nondecreasing sequence of α-finite sets. By hypothesis, $\gamma - M$ has order type less than α^*. Hence by Lemma 3.3, there is a σ such that $\gamma - M^\sigma$ has order type less than α^*. Also, $\gamma - M^\sigma$ is α-finite; hence $\gamma - M^\sigma$ can be put into α-finite one-one correspondence with some ordinal less than α^*. Hence any Π_1 subset of $\gamma - M^\sigma$ is α-finite. In particular, S is α-finite. □

THEOREM 4.3. *Suppose α^* is less than α. Let S be a Π_1 subset of α^* which is unbounded in α^* and cannot be split by a Π_1 set into two parts each unbounded in α^*. Then for each $\mu < \alpha^*$, there is a Σ_3 function $f: \mu \to \alpha^*$ such that $\{\xi < u \mid f(\xi) < \gamma\}$ is finite for each $\gamma < \alpha^*$.*

PROOF. Obviously S is not α-finite. Hence by 4.2, S has order type α^*. Hence no final segment of S has order type less than α^*. Proceed as in the proof of Theorem 3.4. □

The next theorem says that maximal α-r.e. subsets of α^* do not exist for uncountable admissible ordinals α.

THEOREM 4.4. *Suppose $\alpha > \alpha^* \geqq \omega_1^L$. Then every Π_1 subset of α^* unbounded in α^* can be split by a Π_1 set into two parts each unbounded in α^*.*

PROOF. If $\alpha^* = \omega_1^L$, apply Theorem 4.3 with $\mu = \omega$. If $\alpha^* > \omega_1^L$, apply Theorem 4.3 with $\mu = \omega_1^L$. □

5. R-maximal sets

In ordinary recursion theory, an r-maximal set is defined as an r.e. set whose complement is infinite but cannot be split by a recursive set into two infinite parts. It is known that there exist r-maximal sets which are not maximal. (This result is due to A. H. Lachlan and R. W. Robinson, independently. See Rogers [5. pp. 252–3].) This suggests that we try to study r-maximal sets in α-recursion theory.

Unfortunately, Theorems 3.4 and 4.3 say nothing about r-maximal sets. It is unknown, for instance, whether there is an uncountable admissible ordinal α such that r-maximal α-r.e. sets exist. Theorem 3.2 together with the following theorem gives some fragmentary information.

THEOREM 5.1. *Assume that α^* is not a limit of α-cardinals, and that the Σ_3 cofinality of α is α^*. Then:*

i) *every unbounded Σ_2 set can be split by an α-recursive set into two unbounded parts.*

ii) *every Σ_2 subset of α^* unbounded in α^* can be split by an α-recursive set into two parts each unbounded in α^*.*

PROOF. We are assuming that α^* is not a limit of α-cardinals. Let β be the largest α-cardinal less than α^*. By Cantor's theorem inside M_α, there is a one-one α-recursive function H from α into the α-finite subsets of β.

Let S be an unbounded Σ_2 set which cannot be split by an α-recursive set into two unbounded parts. For each $\nu < \beta$, define

$$B_\nu = \{\sigma < \alpha \mid \nu \in H_\sigma\}.$$

Then B_ν is α-recursive. Hence either $S \cap B_\nu$ or $S - B_\nu$ is bounded. The relation

$$S \cap B_\nu \subseteq \gamma \vee S - B_\nu \subseteq \gamma$$

is clearly Π_2. Hence by Jensen's Theorem 1.1, there is a Σ_3 function $g: \beta \to \alpha$ such that

$$S \cap B_\nu \subseteq g(\nu) \vee S - B_\nu \subseteq g(\nu)$$

for each $\nu < \beta$. We are assuming that the Σ_3 cofinality of α is greater than β. Hence $g''\beta$ is bounded. Let γ, δ be elements of S such that $g''\beta \subseteq \gamma < \delta$. Then $H_\gamma = H_\delta$. This contradicts the fact that H is one-one.

We have just proved (i). The proof of (ii) is similar, noting that α^* is equal to its own Σ_3 cofinality. □

6. Open questions

We list some open questions which have been partially answered by the results of the present paper.

1. For which admissible ordinals α does there exist an unbounded Π_1 set which cannot be split by a Π_1 set into two unbounded parts? †

2. For which admissible ordinals α does there exist an unbounded Π_1 or Σ_2 set which cannot be split by an α-recursive set into two unbounded parts?

3. There are similar questions for subsets of α^*. In particular, are there an admissible ordinal α such that $\omega < \alpha^* < \alpha$ and a Π_1 subset of α^* unbounded in α^* which cannot be split by a Π_1 set into two parts each unbounded in α^*?

We say that a set $C \subseteq \alpha$ is *cohesive* if C is unbounded but cannot be split by a

† The first author has recently studied various definitions of maximality and has obtained a necessary and sufficient condition for the existence of a maximal α-r.e. set for each such definition. The results will appear in a paper entitled "Maximal α-r.e. sets" and will provide an answer to Question 1.

Π_1 set into two unbounded parts. It follows from the proofs of Theorem 3.2 and 5.1 that if $V = L$ and α is either a successor cardinal or a limit cardinal with Σ_2 cofinality $< \alpha$, then cohesive subsets of α do not exist. The standard construction of a cohesive subset of ω (see [5, pp. 231–232]) generalizes to show that if α is a weakly compact cardinal of L, then α has a cohesive subset. (We originally noted this for α measurable, and E. Fisher observed that weak compactness in L suffices.)

4. Which admissible ordinals have cohesive subsets? In particular, can it be proved in ZF that there is a cardinal α of L such that $\alpha > \omega$ and α has a cohesive subset ? †

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Department of Mathematics
Yale University

† R. Shore has recently answered the last part of Question 4 in the affirmative.